Application of the Laplace Transform for the Present Value

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Abstract
The aim of the article is to demonstrate the use of the Laplace transform to the evaluation of consol’s present value under different streams of returns (dividends). Consol’s (or perpetual bonds, perpetuities) are bonds with no maturity, which means that interest is paid to a bondholder perpetually (usually annually) forever. The present value of a consol, when constant interest is paid, is simply a ratio of this interest and interest rate. However, when interest payments (stream of returns to a bondholder) change in time the evaluation of consol’s present value is more complicated. In the first part of this article it is shown that the Laplace transform of a stream of returns can be used to evaluate present value of a consol under assumption of continuous compounding; and moreover, with the inverse Laplace transform an unknown stream of returns can be reconstructed from the known consol’s present value. In the second part of the paper, the use of the Laplace transform for the present value evaluation is illustrated by examples.

Keywords: Consol, future value, Laplace transforms, present value, stream of returns

I. Introduction

During the past few decades, methods based on integral transforms, in particular, the Laplace transforms, are being increasingly employed in mathematics, physics, mechanics and other engineering sciences. Laplace transforms have a wide variety of applications in the solution of differential, integral and difference equations. It is much less used in financial engineering. One reason is technical: not many people know that all that they need to do is to make simple calculations in the Laplace domain. The value of a consol is equal to the ratio of interest and interest rate. Therefore, consol’s value rises when interest rates decreases and vice versa.

The article is organized as follows:
• In Section 1 the Laplace transform and its basic properties are introduced,
• In Section 2 the general properties of the Laplace transformation.
• In Section 3 the present and the future value of payments is briefly described
• In Section 4 provides the evaluation of the present value of consol’s as well a solution to the inverse problem with unknown streams of return.

I. Basic Definitions and Results

Let \( f(t) \) be a function defined on \([0, \infty)\). The Laplace transform of \( f(t) \) is a new function defined as

\[
L \{f(s)\} = \int_{0}^{\infty} e^{-st} f(t) dt
\]

The domain of \( L \{f\} \) is the set of \( s \in \mathbb{R} \), such that the improper integral converges.

- We will say that the function \( f(t) \) has an exponential order at infinity if and only if, there exist \( \alpha \) and \( M \) such that \( |f(t)| \leq Me^{\alpha t} \), \( \forall t \geq 0 \).

Existence of Laplace transform
Let \( f(t) \) be a function piecewise continuous on \([0, A]\) (for every \( A > 0 \)) and have an exponential order at infinity with \( |f(t)| \leq Me^{\alpha t} \). Then, the Laplace transform \( L \{f\} \) is defined for \( s > \alpha \), that is \( s > \alpha \in \text{Domain } (L \{f\}) \).

Uniqueness of Laplace transform
Let \( f(t) \), and \( g(t) \), be two functions piecewise continuous with an exponential order at infinity. Assume that

\[
L \{f\} = L \{g\}
\]

then \( f(t) = g(t) \) for \( t \in [0, B] \), for every \( B > 0 \), except maybe for a finite set of points.

- If \( L \{f\} = f(s) \), then \( e^{st} f \) \( \in L \{s \rightarrow c\} \)

- Suppose that \( f(t) \), and its derivatives \( f^k(t) \), for \( k \in [1, n] \), are piecewise continuous and have an exponential order at infinity. Then we have
L \{ \frac{d^n}{dt^n} f(t) \} = s^n L \{ f(t) \} s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \ldots \ldots - f^{(n-1)}(0)

This is a very important formula because of its use in differential equations.

- Let \( f(t) \) be a function piecewise continuous on \([0, A]\) (for every \( A>0 \)) and have an exponential order at infinity. Then we have
  \[ L \{ t^n f(t) \} = (-1)^n f^n(s) \]
  where \( f^n \) is the derivative of order \( n \) of the function \( F \).

- Let \( f(t) \) be a function piecewise continuous on \([0, A]\) (for every \( A>0 \)) and have an exponential order at infinity. Suppose that the limit \( \lim_{t \to 0^+} \frac{f(t)}{t} \), is finite. Then we have
  \[ L \{ \frac{f(t)}{t} \} = \int_s^\infty f(s) \, ds \]

II. General Properties of the Laplace Transformation

Beyond the simple intuitive appeal of each of the rules in Table I, the collection is interesting in the sense that all six expressions can be derived from a single property of the Laplace transformation. That unifying rule is identified in Table II (line 8) along with other properties of the Laplace transformation that appear to have particular significance for finance. Line 1 states that the Laplace transformation is a linear operator. Line 2 shows that scaling a cash flow by a geometric growth term is equivalent to a corresponding reduction in the rate of discount. Both rules are readily apparent from the definition of the Laplace transformation as the integral of an exponentially weighted function \( \mathcal{L}(r) = \int_0^\infty e^{-rt} C(t) \, dt \). Line 3 shows the effect of scaling a cash flow by an arithmetic growth term. Readers who are familiar with the Hicks/Macaulay measure of duration (time weighted present value) should recognize the link to interest rate elasticity that is implied by this rule. To confirm the result, recall that the derivative of the exponential function, \( \exp(-rt) \), taken with respect to \( r \), is simply the function itself scaled by \(-t\). The rule for division by the time index (line 4) is a corollary to line 3 that follows from Leibnitz’s rule for the derivative of a definite integral taken with respect to its lower bound. Line 5 applies the change-of-variable theorem of integral calculus and is particularly useful for evaluating cash flows with

<table>
<thead>
<tr>
<th>Cash flow</th>
<th>Transform</th>
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<tbody>
<tr>
<td>( aC(t) + bD(t) )</td>
<td>( aV(r) + bW(r) )</td>
</tr>
<tr>
<td>( \frac{d}{dt} C(t) )</td>
<td>( V(r) )</td>
</tr>
<tr>
<td>( C(a+bt) ), for ( t \geq a/b )</td>
<td>( \int_c^s \mathcal{L}(r) , dx )</td>
</tr>
<tr>
<td>( 0 ), for ( t &lt; a/b )</td>
<td>( e^{\frac{s-a}{b}} \mathcal{L}(r) )</td>
</tr>
<tr>
<td>( C(t) ), for ( t = k )</td>
<td>( \mathcal{L}(r) - \mathcal{L}(\frac{k}{r}) )</td>
</tr>
<tr>
<td>( 0 ), for ( t \neq k )</td>
<td>( V(\frac{r}{k}) )</td>
</tr>
<tr>
<td>( C(t) ), for ( t \leq k )</td>
<td>( \mathcal{L}(\frac{k}{r}) )</td>
</tr>
<tr>
<td>( 0 ), for ( t &gt; k )</td>
<td>( \mathcal{L}(\frac{k}{r}) )</td>
</tr>
</tbody>
</table>

* In the table, \( a \) and \( b \) are arbitrary constants and \( V(r) \) and \( W(r) \) are the transforms of the cash flows \( C(t) \) and \( D(t) \), respectively.
altered time schedules (accelerated or deferred.) In addition, rule 5 can be used in conjunction with the trivial rule for the transform of a single payment (line 6) to evaluate flows with finite lives as indicated on line 7. Line 8 identifies a fundamental linear relationship between Laplace transforms for cash flows and their time derivatives. This property is worthy of special note for two reasons. First, the proof is nontrivial, and that alone sets it apart from the other rules in Table II. Second, the property is a generalization of the customary procedure for solving the present value equation by applying the rule for summing (or integrating) geometric series. To confirm the time-derivative property, note that integration by parts implies that:

\[ e^{-rt} C(t) = -r \int e^{-rt} C(t) dt \]

Rule 8 follows immediately from Equation (1) when we evaluate the integral over the relevant range for the Laplace transform \([0, \infty]\) and impose a standard assumption in present value problems that the marginal present value of the cash flow, \(e^{-rt}\), vanishes as \(t\) gets large. Line 9 is a corollary to property 8 that follows from Leibnitz’s rule for the derivative of a definite integral taken with respect to its upper bound.

III. The present and the future value

The future value (FV) of the present value (PV) of a payment after \(n\) periods of time (usually years) with the effective interest rate \(r\) is given as:

\[ FV = PV (1 + r)^n \]

For the present value of a payment we obtain:

\[ PV = FV (1 + r)^{-n} \]

The future value of the present value with periodic compounding of interest, where \(n\) is the number of times interest is compounded per year and \(t\) denotes the number of years, is given as:

\[ FV = PV (1 + \frac{r}{n})^{nt} \]

And for the present value we obtain:

\[ PV = FV (1 + \frac{r}{n})^{-nt} \]

Suppose that compounding is continuous \((n \to \infty)\). With the use of well-known limit for the Euler’s number: \(\lim_{n \to \infty} (1 + \frac{r}{n})^n = e^r\) see \(T(p) = \int_0^b f(t)k(p,t) dt\) we transform (3) and (4) into the following formulas for the future value and the present value respectively:

\[ FV = PV e^{rt} \]

\[ PV = FV e^{-rt} \]

Now, let’s consider a stream of returns (or payments) instead of a single return. This situation might be relevant for large companies with continuous returns such as chain stores, electricity providers, Internet commerce and others. To achieve the present value of a stream of returns over a period of \(M\) years, we divide the interval \([0, M]\) into \(n\) subintervals such that \(0 = t_1 < t_2 < t_3 < ... < t_n = M\). We assume that in each interval \([t_{i-1}, t_i]\) of a length \(\Delta t_i = t_i - t_{i-1}\) a single return is realized. Then the future and the present value of the stream of returns \(S(t_i)\) with the interest \(r\) are given as:

\[ FV = \sum_{i=1}^{n} S(t_i) e^{rt_i} \Delta t_i \]

\[ PV = \sum_{i=1}^{n} S(t_i) e^{-rt_i} \Delta t_i \]

Making intervals \([t_{i-1}, t_i]\) infinitesimally small \((n \to \infty)\). Yields the following integral formulas:

\[ FV = \int_0^M S(t) e^{rt} dt \]

\[ PV = \int_0^M S(t) e^{-rt} dt \]

In the case of consol a periodic interest is paid virtually forever, so the value \(M\) is infinite \((M \to \infty)\) and from (9) to (10) we obtain:

\[ FV = \int_0^\infty S(t) e^{rt} dt \]

\[ PV = \int_0^\infty S(t) e^{-rt} dt \]

By the comparison of \(L \{ f(t) \} = \int_0^\infty f(t) e^{-pt} dt\) and (12) we see that to obtain the present value of a stream of returns \(S(t)\) we have to find the Laplace transform of a function \(S(t)\), where \(p\) in Definition 2 corresponds to the interest rate \(r\). Moreover, it is possible to recover unknown stream of returns from a known present value through the inverse Laplace transform (or we can simply use results provided in Table 1 ‘backwards’). The use of the Laplace transform for the consol present value evaluation is illustrated in the next Section.
IV. The evaluation of consol's present value by the Laplace transform

In this Section the use of the Laplace transform for the evaluation of consol’s present value with different streams of returns is demonstrated on several examples.

**Example 1.** Let the stream of returns \( S(t) \) be a constant value \( k \) and interest rate be \( r \). What is the present value of a consol?

From (12) and \( L\{1\} = \frac{1}{s} \) we get:

\[
PV = \int_{0}^{\infty} k e^{-rt} \, dt = \frac{k}{r}
\]

The ratio \( k/r \) is a well-known formula for the present value of a consol. For example, if \( k = 1 \) $ per year and \( r = 2\% \) (0.02), then \( PV = 50 \) $.

**Example 2.** Let the stream of returns be linearly growing, \( S(t) = t \) with interest rate \( r \). What is the present value of a consol?

From (12) and \( L\{t^n\} = \frac{n!}{s^{n+1}} \) we get:

\[
PV = \int_{0}^{\infty} t e^{-rt} \, dt = \frac{1}{r^2}
\]

For example, if \( r = 3 \% \), then \( PV = 1111.1 \) $.

**Example 3.** Let \( r \) be interest rate and let the stream of returns \( S(t) \) be exponentially growing at a rate \( q \), \( q < r \). What is the present value of a consol?

From (12) and \( L\{e^{at}\} = \frac{1}{s-a} \) we get

\[
PV = \int_{0}^{\infty} e^{qt} e^{-rt} \, dt = \frac{1}{r-q}
\]

For example, if \( q = 1 \% \) and \( r = 2 \% \), then \( PV = 100 \) $.

**Example 4.** The present value of a consol with the interest rate \( r \) is \( \frac{1}{r} \). Find the stream of returns \( S(t) \).

First, we transform a given fraction in two partial fractions:

\[
\frac{1}{r^2} = \frac{1}{r} + \frac{1}{r-1}
\]

Then we use \( L\{1\} = \frac{1}{s} \) and \( L\{e^{at}\} = \frac{1}{s-a} \) to find the original to these images:

\[
L^{-1}\left\{\frac{1}{r} \right\} = 1
\]

And

\[
L^{-1}\left\{-\frac{1}{r-1} \right\} = e^t
\]

With the use of (linearity of the Laplace transform) we finally obtain the stream of returns \( S(t) = e^t - 1 \) $/year. Finally, Theorem 1 ensures this is the only solution to the problem.

**Conclusion:**

The aim of the article is to demonstrate the application of the Laplace transform in economics, namely in the evaluation of consol’s present value when different streams of returns (constant, linearly or exponentially growing, etc.) are involved. The Laplace transform allows easy computation of consol’s present value from the known stream of returns, but it also enables to reconstruct unknown pattern of returns from the present value through inverse Laplace transformation. In the last section of the paper, the use of the Laplace transform is illustrated on several examples.

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