Ratio cum Median Based Modified Ratio Estimator for the Estimation of Finite Population Mean with Known Skewness and Kurtosis

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Abstract

The present paper deals with some class of ratio cum median based modified ratio estimators for estimating the finite population mean with known parameters of the auxiliary variable such as skewness, kurtosis and their linear combinations. The efficiencies of proposed estimators are assessed with that of simple random sampling without replacement (SRSWOR) sample mean and ratio estimator in terms of their variances and or mean squared errors. The aforesaid efficiency comparison is illustrated with the help of a natural population available in the literature. Further it is observed that the proposed estimators perform better than the existing estimators considered in this study.

AMS Subject Classification- 62D05

Keywords: Auxiliary variable; Bias; Mean squared error; Natural population; Percentage relative efficiency; Simple random sampling.

1. INTRODUCTION

Consider a finite population \( U = \{U_1, U_2, ..., U_N\} \) of \( N \) distinct and identifiable units. Let \( Y(X) \) denote the study (auxiliary) variable taking values \( Y_i(X_i) \) measured on each \( U_i, i = 1, 2, ..., N \). Here the main objective of this study is to estimate the parameter, population mean \( \bar{Y} \) of the study variable by selecting a random sample of size \( n \) from the population \( U \) with some desirable properties like unbiasedness, minimum variance etc. Hence one can use the simple random sampling without replacement (SRSWOR) technique, which provides an unbiased estimator for the population mean. Suppose that, if there exists an auxiliary variable and is positively correlated with that of the study variable one can use ratio method of estimation to improve the efficiency of SRSWOR sample mean. For a detailed discussion on ratio estimator, one can refer to Cochran (1977) and Murthy (1967). Further improvements on ratio estimator are achieved by making use of the known parameters of the auxiliary variable with known correlation coefficient, coefficient of variation, Skewness, Kurtosis, Quartiles etc. and the resulting estimators are called modified ratio estimators. See for example Adepoju and Shittu (2013), Das and Tripathi (1978), Diana, Giordan and Perri (2011), Gupta and Shabbir(2008), Kadilar and Cingi (2004), Kadilar and Cingi (2006a, 2006 B), Koyuncu (2012), Koyuncu and Kadilar (2009), Shittu and Adepoju (2013), Singh and Agnihotri (2008), Singh and Tailor(2003), Sisodia and Dwivedi (1981), Subramani and Kumarapandian (2012a, 2012b) and the references cited there in.

The median based ratio estimator is another estimator suggested by Subramani (2013) that makes use of the population median of the study variable\( Y \). It is to be noted that the median based ratio estimator outperforms SRSWOR sample mean, ratio estimator and also the linear regression estimator. By extending the median based ratio estimator median based modified ratio estimators are studied by Subramani and Prabavathy (2014a, 2014b, 2015).

However, no attempt has been made using the linear combinations of modified ratio estimator and median based modified ratio estimator to improve the efficiency of the modified ratio estimators. Hence, an attempt is made in this paper to introduce some ratio cum median based modified ratio estimators using known parameters of the auxiliary variable such as coefficient of kurtosis, coefficient of skewness and their linear combinations.
combinations. Before discussing about the proposed estimators, we discuss some of the existing estimators together with the notations to be used and are as follows.

1.1. Notations to be used

- $N$ – Population size
- $n$ – Sample size
- $f = n/N$, Sampling fraction
- $\delta = \frac{1-f}{n}$, finite population correction
- $\bar{X}, \bar{Y}$ – Population means
- $\bar{x}, \bar{y}$ – Sample means
- $S_X, S_Y$ – Population standard deviations
- $S_{xy}$ – Population covariance between $X$ and $Y$
- $C_X(C_Y)$ – Co-efficient of variation of $X$ ($Y$)
- $\rho = \frac{S_{xy}}{S_X S_Y}$ – Co-efficient of correlation between $X$ and $Y$
- $\beta_1 = \frac{\mu_3}{\mu_2^2}$, Skewness of the auxiliary variable
- $\beta_2 = \frac{\mu_4}{\mu_2^2}$, Kurtosis of the auxiliary variable where $\mu_r = \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X})^r$
- $M$ (m) – Population (sample) Median of the study variable
- $B(.)$ – Bias of the estimator
- $MSE(.)$ – Mean squared error of the estimator
- $V(.)$ – Variance of the estimator
- $\bar{y}$ – Simple random sampling without replacement (SRSWOR) sample mean
- $\bar{Y}_R$ – Ratio estimator
- $\bar{Y}_M$ – Median Based Ratio Estimator
- $\bar{Y}_{pj}$ – $j$th Proposed median based modified ratio estimator of $\bar{Y}$

1.2. Existing Estimators

In case of SRSWOR, the sample mean $\bar{y}$ is used to estimate population mean $\bar{Y}$ which is an unbiased estimator. The SRSWOR sample mean together with its variance is given below:

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$$  \hspace{1cm} (1)

$$V(\bar{y}) = \frac{(1-f)}{n} S_y^2$$  \hspace{1cm} (2)

where $f = \frac{n}{N}$, $S_y^2 = \frac{1}{N-1} \sum_{i=1}^{N} (Y_i - \bar{Y})^2$

The ratio estimator for estimating the population mean $\bar{Y}$ of the study variable $Y$ is defined as

$$\bar{Y}_R = \frac{\bar{y}}{\bar{x}} \bar{X}$$  \hspace{1cm} (3)

The mean squared error of $\bar{Y}_R$ is given below:

$$MSE(\bar{Y}_R) = \bar{Y}^2 \left( C_{yy} + C'_{xx} - 2C_{yx} \right)$$  \hspace{1cm} (4)

where $C_{yy} = \frac{V(\bar{y})}{\bar{y}^2}$, $C'_{xx} = \frac{V(\bar{x})}{\bar{x}^2}$, $C_{yx} = \frac{\text{cov}(\bar{y}, \bar{x})}{\bar{y}\bar{x}}$
2. PROPOSED ESTIMATORS

In this section a class of ratio cum median based modified ratio estimators with known parameters of the auxiliary variable like coefficient of Kurtosis, coefficient of skewness and their linear combinations has been proposed. The estimators together with their mean squared errors are given below:

The proposed estimator with the population parameter $T_i$ is

$$\hat{Y}_{p_i} = \bar{y} \left( \alpha_1 \left( \frac{M + T_i}{m + T_i} \right) + \alpha_2 \left( \frac{X + T_i}{X + T_i} \right) \right)$$

where $\alpha_1 + \alpha_2 = 1$, $i = 1,2,3,4$.

The mean squared error of proposed estimator is given as

$$\text{MSE}\left(\hat{Y}_{p_i}\right) = \bar{Y}^2 \left\{ C'_{yy} + \alpha_1^2 \theta_i^2 C'_{mm} + \alpha_2^2 \varphi_i^2 C'_{xx} - 2\alpha_1 \theta_i C'_{ym} - 2\alpha_2 \varphi_i C'_{yx} + 2\alpha_1 \alpha_2 \theta_i \varphi_i C'_{xm} \right\}$$

where $\theta_i = \frac{M}{M + T_i}$, $\varphi_i = \frac{\bar{X}}{X + T_i}$, $C_{xm} = \frac{\text{Cov}(\bar{X}, m)}{M \bar{X}}$, $T_1 = \beta_1$, $T_2 = \beta_2$, $T_3 = \beta_1/\beta_2$, $T_4 = \beta_2/\beta_1$.

The detailed derivation of the above expression of the mean squared error is given in the Appendix.

3. EFFICIENCY COMPARISON

In this section, the efficiencies of proposed estimators are assessed with that of SRSWOR sample mean and ratio estimator in terms of variance/mean squared error. The results are as follows.

3.1. Comparison with that of SRSWOR sample mean

Comparing (6) and (2), it is noticed that the proposed estimators perform better than the SRSWOR sample mean if $\text{MSE}(\hat{Y}_{p_i}) \leq \text{V}(\bar{Y})$ i.e.,

$$\alpha_1^2 \theta_i^2 C'_{mm} + \alpha_2^2 \varphi_i^2 C'_{xx} + 2\alpha_1 \alpha_2 \theta_i \varphi_i C'_{xm} \leq 2(\alpha_1 \theta_i C'_{ym} + \alpha_2 \varphi_i C'_{yx})$$

(7)

3.2. Comparison with that of Ratio Estimator

Comparing (6) and (4), it is noticed that the proposed estimators perform better than the ratio estimator if $\text{MSE}(\hat{Y}_{p_i}) \leq \text{MSE}(\hat{Y}_R)$ i.e.,

$$\alpha_1^2 \theta_i^2 C'_{mm} + (\alpha_2^2 \varphi_i^2 - 1) C'_{xx} + 2\alpha_1 \alpha_2 \theta_i \varphi_i C'_{xm} \leq 2\left[\alpha_1 \theta_i C'_{ym} + (\alpha_2 \varphi_i - 1) C'_{yx}\right]$$

(8)

3.3. Numerical Comparison

In the sections 3.1 and 3.2, the conditions for the efficiency of proposed estimators with that of existing estimators have been derived algebraically. To support it by means of numerical comparison, a natural population from Daroga Singh and Chaudhary (1986, page.177) has been considered.

Population Description

$X =$ Area under Wheat in 1971 and $Y =$ Area under Wheat in 1974

The population parameters computed for the above population is given below:

$$N = 34 \quad n = 3 \quad \bar{Y} = 856.4118$$

$$\rho = 0.4491 \quad M = 767.5 \quad \bar{X} = 208.8824$$

$$C_x = 0.7205 \quad \beta_2 = 2.9123 \quad \beta_1 = 0.8732$$

The variance/mean squared error of the existing and proposed estimators at different values of $\alpha_1$ and $\alpha_2$ are given in the following table.
Table 1: Mean Squared Errors for different values of $\alpha_{1}$ and $\alpha_{2}$

<table>
<thead>
<tr>
<th>Proposed Estimators</th>
<th>$\hat{Y}_{p_1}$</th>
<th>$\hat{Y}_{p_2}$</th>
<th>$\hat{Y}_{p_3}$</th>
<th>$\hat{Y}_{p_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{1}$</td>
<td>$\alpha_{2}$</td>
<td>$\hat{Y}_{p_1}$</td>
<td>$\hat{Y}_{p_2}$</td>
<td>$\hat{Y}_{p_3}$</td>
</tr>
<tr>
<td>0</td>
<td>1.0</td>
<td>155132.31</td>
<td>154117.20</td>
<td>155425.20</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>141799.50</td>
<td>140946.34</td>
<td>142045.71</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>129944.85</td>
<td>129237.21</td>
<td>130149.08</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7</td>
<td>119568.33</td>
<td>118989.81</td>
<td>119735.32</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>110669.96</td>
<td>110204.14</td>
<td>110804.41</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>103249.74</td>
<td>102880.20</td>
<td>103356.36</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>97307.66</td>
<td>97017.99</td>
<td>97391.17</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3</td>
<td>92843.73</td>
<td>92617.52</td>
<td>92908.84</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>89857.94</td>
<td>89678.77</td>
<td>89909.37</td>
</tr>
<tr>
<td>0.9</td>
<td>0.1</td>
<td>88350.29</td>
<td>88201.76</td>
<td>88392.76</td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>88320.79</td>
<td>88186.48</td>
<td>88359.00</td>
</tr>
</tbody>
</table>

From Table 1, it is observed that the proposed estimators have less mean squared errors than the SRSWOR sample mean and ratio estimator.

The percentage relative efficiencies (PRE) of the proposed estimators with respect to the existing estimators are obtained by using the formula $\text{PRE}(e, p) = \frac{\text{MSE}(e)}{\text{MSE}(p)} \times 100$ and are given in the following table:

Table 2: PRE of proposed estimators with respect to SRSWOR sample mean

<table>
<thead>
<tr>
<th>$\alpha_{1}$</th>
<th>$\alpha_{2}$</th>
<th>$\hat{Y}_{p_1}$</th>
<th>$\hat{Y}_{p_2}$</th>
<th>$\hat{Y}_{p_3}$</th>
<th>$\hat{Y}_{p_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>105.30</td>
<td>105.99</td>
<td>105.10</td>
<td>106.14</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>115.20</td>
<td>115.90</td>
<td>115.00</td>
<td>116.04</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>125.71</td>
<td>126.40</td>
<td>125.51</td>
<td>126.54</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7</td>
<td>136.62</td>
<td>137.29</td>
<td>136.43</td>
<td>137.42</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>147.61</td>
<td>148.23</td>
<td>147.43</td>
<td>148.36</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>158.21</td>
<td>158.78</td>
<td>158.05</td>
<td>158.90</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>167.88</td>
<td>168.38</td>
<td>167.73</td>
<td>168.48</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3</td>
<td>175.95</td>
<td>176.38</td>
<td>175.82</td>
<td>176.46</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>181.79</td>
<td>182.16</td>
<td>181.69</td>
<td>182.23</td>
</tr>
<tr>
<td>0.9</td>
<td>0.1</td>
<td>184.90</td>
<td>185.21</td>
<td>184.81</td>
<td>185.27</td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>184.96</td>
<td>185.24</td>
<td>184.88</td>
<td>185.30</td>
</tr>
</tbody>
</table>
Table 3: PRE of proposed estimators with respect to Ratio Estimator

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\hat{Y}_{P_1}$</th>
<th>$\hat{Y}_{P_2}$</th>
<th>$\hat{Y}_{P_3}$</th>
<th>$\hat{Y}_{P_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>100.29</td>
<td>100.95</td>
<td>100.10</td>
<td>101.08</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>109.72</td>
<td>110.38</td>
<td>109.53</td>
<td>110.52</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>119.73</td>
<td>120.38</td>
<td>119.54</td>
<td>120.52</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7</td>
<td>130.12</td>
<td>130.75</td>
<td>129.94</td>
<td>130.88</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>140.58</td>
<td>141.17</td>
<td>140.41</td>
<td>141.29</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>150.68</td>
<td>151.22</td>
<td>150.53</td>
<td>151.33</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>159.88</td>
<td>160.36</td>
<td>159.75</td>
<td>160.46</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3</td>
<td>167.57</td>
<td>167.98</td>
<td>167.45</td>
<td>168.06</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>173.14</td>
<td>173.49</td>
<td>173.04</td>
<td>173.56</td>
</tr>
<tr>
<td>0.9</td>
<td>0.1</td>
<td>176.09</td>
<td>176.39</td>
<td>176.01</td>
<td>176.45</td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>176.15</td>
<td>176.42</td>
<td>176.08</td>
<td>176.48</td>
</tr>
</tbody>
</table>

From Table 2 and 3, it is observed that the PRE values of the proposed estimators with respect to SRSWOR sample mean and ratio estimator are greater than 100 and hence we conclude that the proposed estimators have greater efficiency. In fact the PREs are ranging from 105.30 to 185.30 for the case of SRSWOR sample mean where as it is 100.10 to 176.48 for the case of ratio estimator.

4. CONCLUSION

The present paper deals with some ratio cum median based modified ratio estimators with known parameters such as skewness and kurtosis of the auxiliary variable. The efficiencies of the proposed estimators are assessed with that of SRSWOR sample mean and ratio estimator algebraically as well as numerically. Further it is observed that the PRE of proposed estimators with respect to the existing estimators are greater than 100. Hence the proposed estimators may be recommended for the use of practical applications.

REFERENCES

APPENDIX-A

DERIVATION OF BIAS AND MEAN SQUARED ERROR OF RATIO CUM MEDIAN BASED MODIFIED RATIO ESTIMATORS

The derivation of the bias and mean squared error of the proposed median based modified ratio estimator $\hat{\psi}_{pi}, i = 1, 2, 3, 4$ are given below:

Consider $\hat{\psi}_{pi} = \bar{Y} \left[ \alpha_1 \left( \frac{m+T_i}{m+T_i} \right) + \alpha_2 \left( \frac{x+T_i}{x+T_i} \right) \right]$ where $\alpha_1 + \alpha_2 = 1 \quad (A1)$

where $\bar{y}$ is the SRSWOR sample mean of the study variable $Y$

mis the sample median of the study variable $Y$

Mis the population median of the study variable $Y$

$T_i$ is the parameter or ratio of parameters of the auxiliary variable $X$

Let $e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}}, e_1 = \frac{m-M}{M}$ and $e_2 = \frac{x-\bar{X}}{\bar{X}}$

$\Rightarrow E(e_0) = 0; \ E(e_1) = \frac{M-m}{M} = \frac{B(m)}{M}$ and $E(e_2) = 0$

$\Rightarrow E(e_0^2) = \frac{V(\bar{y})}{\bar{Y}^2}; E(e_1^2) = \frac{V(m)}{M^2}; E(e_2^2) = \frac{V(x)}{\bar{X}^2} \ E(e_0 e_1) = \frac{\text{Cov}(\bar{y}, m)}{\bar{Y} M}$

$E(e_0 e_2) = \frac{\text{Cov}(\bar{y}, x)}{\bar{X} M}; E(e_1 e_2) = \frac{\text{Cov}(x, m)}{\bar{X} M}$

The estimator $\hat{\psi}_{pi}$ can be written in terms of $e_0, e_1$ and $e_2$ as
\[ \hat{\tilde{Y}}_{p_1} = \bar{Y} + e_0 \left\{ \alpha_1 \left( \frac{M + T_i}{M(1 + e_1) + T_i} \right) + \alpha_2 \left( \frac{X + T_i}{X(1 + e_2) + T_i} \right) \right\} \]

\[ \Rightarrow \hat{\tilde{Y}}_{p_1} = \bar{Y} + e_0 \left\{ \alpha_1 \left( \frac{1}{1 + \frac{M}{M + T_i}} \right) e_1 + \alpha_2 \left( \frac{1}{1 + \frac{X}{X + T_i}} \right) e_2 \right\} \]

\[ \Rightarrow \hat{\tilde{Y}}_{p_1} = \bar{Y} + e_0 \left\{ \alpha_1 \left( \frac{1}{1 + \theta_1 e_1} \right) + \alpha_2 \left( \frac{1}{1 + \phi_2 e_2} \right) \right\} \text{; where } \theta_1 = \frac{M}{M + T_i}, \phi_1 = \frac{X}{X + T_i} \]

\[ \Rightarrow \hat{\tilde{Y}}_{p_1} = \bar{Y} + e_0 \left\{ \alpha_1 (1 - \theta_1 e_1 - \theta_1 e_2) + \alpha_2 (1 - \phi_1 e_2 - \phi_1 e_2) \right\} \]

Neglecting the terms of higher order, we have

\[ \hat{\tilde{Y}}_i = \bar{Y} + e_0 \left\{ \alpha_1 (1 - \theta_1 e_1 + \theta_1^2 e_1^2) + \alpha_2 (1 - \phi_1 e_2 + \phi_1^2 e_2^2) \right\} \]

\[ \hat{\tilde{Y}}_i = \bar{Y} \left\{ \alpha_1 (1 + e_0 - \theta_1 e_1 - \theta_1 e_2 + \theta_1^2 e_1^2) + \alpha_2 (1 + e_0 - \phi_1 e_2 - \phi_1 e_0 e_2 + \phi_1^2 e_2^2) \right\} \]

Taking expectations on both sides of (A2), one can have,

\[ E(\hat{\tilde{Y}}_p - \bar{Y}) = E(e_0 - \alpha_1 \theta_1 e_1 + \alpha_1 \theta_1 e_0 e_1 - \alpha_2 \phi_1 e_2 + \alpha_2 \phi_1 e_0 e_2) \]

\[ \Rightarrow E(\hat{\tilde{Y}}_p - \bar{Y}) = E \left\{ \alpha_1 \left( \frac{\theta_1^2 V(m)}{M^2} - \theta_1 \frac{B(m)}{M} - \theta_1 \frac{\text{Cov}(\bar{Y}, m)}{YM} \right) + \alpha_2 \left( \phi_1^2 \frac{V(X)}{X^2} - \phi_1 \frac{\text{Cov}(\bar{Y}, X)}{YX} \right) \right\} \]

\[ \Rightarrow B(\hat{\tilde{Y}}_p) = \bar{Y} \left\{ \alpha_1 \left( \theta_1 C_{mn} - \theta_1 C_{ym} - \theta_1 \frac{B(m)}{M} \right) + \alpha_2 \left( \phi_1 C_{xy} - \phi_1 C_{yx} \right) \right\} \] (A3)

Squaring on both sides of (A2), neglecting the terms of higher order and taking expectation on both sides one can get,

\[ \text{MSE}(\hat{\tilde{Y}}_{p_1}) = E(\hat{\tilde{Y}}_{p_1} - \bar{Y})^2 = \bar{Y}^2 + \alpha_1 \theta_1^2 \frac{V(m)}{M^2} - \alpha_1 \theta_1 \frac{B(m)}{M} - \alpha_1 \theta_1 \frac{\text{Cov}(\bar{Y}, m)}{YM} + \alpha_2 \phi_1^2 \frac{V(X)}{X^2} - \alpha_2 \phi_1 \frac{\text{Cov}(\bar{Y}, X)}{YX} \]

\[ \Rightarrow \text{MSE}(\hat{\tilde{Y}}_{p_1}) = \bar{Y}^2 \left\{ \frac{\text{V}(\bar{Y})}{\text{Y}^2} + \alpha_1 \theta_1^2 \frac{V(m)}{M^2} - 2 \alpha_1 \theta_1 \frac{\text{Cov}(\bar{Y}, m)}{YM} + \alpha_2 \phi_1^2 \frac{V(X)}{X^2} - 2 \alpha_2 \phi_1 \frac{\text{Cov}(\bar{Y}, X)}{YX} \right\} \]

After a little algebra, the mean squared error of \( \hat{\tilde{Y}}_{p_1} \) is obtained as

\[ \text{MSE}(\hat{\tilde{Y}}_{p_1}) = \bar{Y}^2 \left\{ \frac{\text{V}(\bar{Y})}{\text{Y}^2} + \alpha_1 \theta_1^2 \frac{V(m)}{M^2} - 2 \alpha_1 \theta_1 \frac{\text{Cov}(\bar{Y}, m)}{YM} + \alpha_2 \phi_1^2 \frac{V(X)}{X^2} - 2 \alpha_2 \phi_1 \frac{\text{Cov}(\bar{Y}, X)}{YX} + 2 \alpha_1 \alpha_2 \theta_1 \phi_1 \frac{\text{Cov}(m, X)}{MX} \right\} \] (A4)

That is,

\[ \text{MSE}(\hat{\tilde{Y}}_{p_1}) = \bar{Y}^2 \left\{ \alpha_1^2 \theta_2^2 C_{mn} + \alpha_2^2 \phi_1^2 C_{xy} - 2 \alpha_1 \theta_1 C_{ym} - 2 \alpha_2 \phi_1 C_{yx} + 2 \alpha_1 \alpha_2 \theta_1 \phi_1 C_{xm} \right\} \]